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LETTER TO THE EDITOR

Algebraic Bethe ansatz for a class of coupled asymmetric six-vertex free-fermion model

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Abstract

We present an algebraic Bethe ansatz for certain submanifolds of the bilayer vertex models proposed by Shiroishi and Wadati as coupled asymmetric six-vertex free-fermion models. A peculiar feature of our formulation is the presence of a diagonal monodromy matrix element that does not generate unwanted terms. The model contains two free-parameters entering into the Bethe ansatz equations as a pure phase factor.

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The six-vertex model satisfying the free-fermion condition possesses certain special properties which are not common to general integrable vertex models of statistical mechanics [1]. This model provides a three-parameter family of commuting transfer matrices [2, 3] whose Boltzmann weights satisfy generalized forms of the star–triangle relations [4, 5]. The latter property is the basis to generate new exactly solvable vertex models by coupling in an appropriate way a pair of six-vertex free-fermion models [4, 5]. This remarkable possibility was originally formulated by Shastry [4] in the case of a totally symmetric six-vertex free-fermion model. As a consequence of that Shastry was able to rederive the R-matrix of the covering vertex model for the one-dimensional Hubbard chain from a rather promising perspective [4].

More recently, Shiroishi and Wadati [6] extended such an approach to include the asymmetric six-vertex free-fermion model which has resulted in new interacting bilayer vertex systems. To our knowledge, however, the Bethe ansatz solution of such potentially interesting integrable bilayer systems is still an open problem. The purpose of this letter is to make a first step towards solving this problem by formulating the algebraic Bethe ansatz for what we expect to be the simplest branch of the bilayer vertex models discovered by Shiroishi and Wadati [6]. This submanifold is interesting both because it does not contain the covering Hubbard model and due to an unusual feature concerning the quantum inverse scattering method. It turns out that one of the diagonal elements of the monodromy matrix does not produce unwanted terms thanks to a remarkable annihilation property of the eigenvectors. The Bethe ansatz equations are of rational type with an additional phase-shift that encodes all possible free-parameter dependence.

The building blocks of the bilayer vertex model constructed by Shiroishi and Wadati are two trigonometric Felderhof's free-fermion models [2] whose Lax operators are defined by

$$L_{\mathcal{A}i}^{(\alpha)}(\lambda, \gamma_i, \gamma_{i+1}) = \begin{pmatrix} \omega_a^+(\lambda, \gamma_i, \gamma_{i+1}) I_i + \omega_a^-(\lambda, \gamma_i, \gamma_{i+1})\sigma_{\alpha,i}^z & \sigma_{\alpha,i}^- \\ \sigma_{\alpha,i}^+ & \omega_b^+(\lambda, \gamma_i, \gamma_{i+1}) I_i + \omega_b^-(\lambda, \gamma_i, \gamma_{i+1})\sigma_{\alpha,i}^z \end{pmatrix} \tag{1}$$

where $\sigma_{\alpha,i}^{\pm}, \sigma_{\alpha,i}^z$ are two sets $\alpha = \uparrow, \downarrow$ of commuting Pauli matrices and I_i is the identity acting on the site i of a lattice of size N. Here $\mathcal{A}$ denotes the auxiliary space, λ is the spectral parameter and γ_k play the role of colour variables that are attached to each kth site of the lattice. The functions $\omega_a^{\pm}(\lambda, \gamma_i, \gamma_j) = \frac{a_+(\lambda,\gamma_i,\gamma_j) \pm b_+(\lambda,\gamma_i,\gamma_j)}{2}$ and $\omega_b^{\pm}(\lambda, \gamma_i, \gamma_j) = \frac{b_-(\lambda,\gamma_i,\gamma_j) \pm a_-(\lambda,\gamma_i,\gamma_j)}{2}$ where $a_{\pm}(\lambda, \gamma_i, \gamma_j)$ and $b_{\pm}(\lambda, \gamma_i, \gamma_j)$ are the elementary Boltzmann weights parametrized in terms of the spectral parameter λ and the colour variables γ_k by

$$a_{\pm}(\lambda, \gamma_i, \gamma_j) = \cosh(\gamma_i - \gamma_j)\cos(\lambda) \pm \sinh(\gamma_i + \gamma_j)\sin(\lambda) \tag{2}$$

$$b_{\pm}(\lambda, \gamma_i, \gamma_j) = \cosh(\gamma_i + \gamma_j)\sin(\lambda) \mp \sinh(\gamma_i - \gamma_j)\cos(\lambda). \tag{3}$$

The Lax operator of the bilayer vertex model is given by a suitable coupling of two Felderhof's free-fermion models and it can be written as follows

$$\begin{aligned} \mathcal{L}_{\mathcal{A}i}(\lambda_i, \lambda_{i+1}, \gamma_i, \gamma_{i+1}) &= L_{\mathcal{A}i}^{(\uparrow)}(\lambda_i - \lambda_{i+1}, \gamma_i, \gamma_{i+1}) L_{\mathcal{A}i}^{(\downarrow)}(\lambda_i - \lambda_{i+1}, \gamma_i, \gamma_{i+1}) \\ &\quad + f_1 L_{\mathcal{A}i}^{(\uparrow)}(\lambda_i + \lambda_{i+1}, \gamma_i, -\gamma_{i+1})\sigma_{\uparrow,\mathcal{A}}^z L_{\mathcal{A}i}^{(\downarrow)}(\lambda_i + \lambda_{i+1}, \gamma_i, -\gamma_{i+1})\sigma_{\downarrow,\mathcal{A}}^z \\ &\quad + f_2\big[L_{\mathcal{A}i}^{(\uparrow)}(\lambda_i + \lambda_{i+1}, \gamma_i, -\gamma_{i+1})\sigma_{\uparrow,\mathcal{A}}^z L_{\mathcal{A}i}^{(\downarrow)}(\lambda_i - \lambda_{i+1}, \gamma_i, \gamma_{i+1}) \\ &\quad + L_{\mathcal{A}i}^{(\uparrow)}(\lambda_i - \lambda_{i+1}, \gamma_i, \gamma_{i+1}) L_{\mathcal{A}i}^{(\downarrow)}(\lambda_i + \lambda_{i+1}, \gamma_i, -\gamma_{i+1})\sigma_{\downarrow,\mathcal{A}}^z\big]. \end{aligned} \tag{4}$$

The solution found by Shiroishi and Wadati [6] for the couplings $f_{1,2}$ depends on two arbitrary constants denoted by A and B in their work. Here we are going to consider the simplest submanifolds as far as the functional behaviour of couplings is concerned. They occur when one of the constants is null because the other can be rescaled to unity without losing generality. Even in this situation there exist at least two free-parameters since the Lax operator (4) depends on both the difference and the sum of the spectral parameters and colour variables. It is convenient to define the couplings of these submanifolds with the help of the relations $g_{\pm}(\lambda, \gamma) = \frac{b_{\pm}(\lambda,\gamma,0)}{a_{\pm}(\lambda,\gamma,0)}$. In terms of these auxiliary functions they are given by

$$f_1 = \begin{cases} \frac{1}{4}\left[\frac{g_+(\lambda_{i+1},\gamma_{i+1})}{g_+(\lambda_i,\gamma_i)} + \frac{g_-(\lambda_{i+1},\gamma_{i+1})}{g_-(\lambda_i,\gamma_i)} - 2\right] & \text{for} \quad B = 0 \\ \frac{1}{4}\left[\frac{g_+(\lambda_i,\gamma_i)}{g_+(\lambda_{i+1},\gamma_{i+1})} + \frac{g_-(\lambda_i,\gamma_i)}{g_-(\lambda_{i+1},\gamma_{i+1})} - 2\right] & \text{for} \quad A = 0 \end{cases} \tag{5}$$

and

$$f_2 = \begin{cases} \frac{1}{4}\left[\frac{g_+(\lambda_{i+1},\gamma_{i+1})}{g_+(\lambda_i,\gamma_i)} - \frac{g_-(\lambda_{i+1},\gamma_{i+1})}{g_-(\lambda_i,\gamma_i)}\right] & \text{for} \quad B = 0 \\ \frac{1}{4}\left[\frac{g_-(\lambda_i,\gamma_i)}{g_-(\lambda_{i+1},\gamma_{i+1})} - \frac{g_+(\lambda_i,\gamma_i)}{g_+(\lambda_{i+1},\gamma_{i+1})}\right] & \text{for} \quad A = 0. \end{cases} \tag{6}$$

The fact that the L-operator (4) satisfies a coloured version of the Yang–Baxter equation [7] implies that it is possible to define a family of row-to-row commuting transfer matrices $T(\lambda)$ as the trace of the monodromy matrix $\mathcal{T}(\lambda)$ on the auxiliary space of the coupled six-vertex free-fermion models. The monodromy matrix then reads

$$\mathcal{T}(\lambda) = \mathcal{L}_{\mathcal{A}L}(\lambda, \lambda_0, \gamma, \gamma)\mathcal{L}_{\mathcal{A}L-1}(\lambda, \lambda_0, \gamma, \gamma) \cdots \mathcal{L}_{\mathcal{A}1}(\lambda, \lambda_0, \gamma, \gamma). \tag{7}$$

These square lattice vertex models have two free parameters λ_0 and γ and their integrability is guaranteed by the following intertwining relation

$$R(\lambda, \mu)\mathcal{T}(\lambda) \otimes \mathcal{T}(\mu) = \mathcal{T}(\mu) \otimes \mathcal{T}(\mu)R(\lambda, \mu) \tag{8}$$

where the R-matrix is $R(\lambda, \mu) = P_{12}\mathcal{L}_{12}(\lambda, \mu, \gamma, \gamma)$ and P_{12} is the permutation operator.

We now turn to the algebraic Bethe ansatz solution of these vertex models. The technicalities entering the solution of the submanifolds $A = 0$ and $B = 0$ are very similar and these details will be presented only for the former case. The main results, however, such as the eigenvalues of the transfer matrices and corresponding Bethe equations will be given for both of them. An essential ingredient in this approach is the existence of a reference state in which the monodromy matrix acts triangularly [8, 9]. This helps us to identify the elements of the monodromy matrix as particle creation and annihilation fields. Here we denote this state by $|0\rangle$ and choose it as the standard ferromagnetic vacuum where all the spins are in the up eigenstate of $\sigma^z_{\alpha,i}$. The action of the monodromy matrix in this state yields the relation

$$\mathcal{T}(\lambda)|0\rangle = \begin{pmatrix} [\omega_1(\lambda)]^N & \ddagger & \ddagger & \ddagger \\ 0 & [\omega_2(\lambda)]^N & 0 & \ddagger \\ 0 & 0 & [\omega_2(\lambda)]^N & \ddagger \\ 0 & 0 & 0 & [\omega_3(\lambda)]^N \end{pmatrix} |0\rangle \tag{9}$$

where the symbol $\ddagger$ denotes non-null values and the functions $\omega_1(\lambda)$, $\omega_2(\lambda)$ and $\omega_3(\lambda)$ are given by

$$\omega_1(\lambda) = a_+^2(\lambda - \lambda_0, \gamma, \gamma) + a_+(\lambda + \lambda_0, \gamma, -\gamma)[f_1 a_+(\lambda + \lambda_0, \gamma, -\gamma) + 2 f_2 a_+(\lambda - \lambda_0, \gamma, \gamma)] \tag{10}$$

$$\begin{aligned} \omega_2(\lambda) = {} & a_+(\lambda - \lambda_0, \gamma, \gamma)b_-(\lambda - \lambda_0, \gamma, \gamma) + f_1 a_+(\lambda + \lambda_0, \gamma, -\gamma)b_-(\lambda + \lambda_0, \gamma, -\gamma) \\ & \times f_2[a_+(\lambda + \lambda_0, \gamma, -\gamma)b_-(\lambda - \lambda_0, \gamma, \gamma) \\ & + a_+(\lambda - \lambda_0, \gamma, \gamma)b_-(\lambda + \lambda_0, \gamma, -\gamma)] \end{aligned} \tag{11}$$

$$\begin{aligned} \omega_3(\lambda) = {} & b_-^2(\lambda - \lambda_0, \gamma, \gamma) + b_-(\lambda + \lambda_0, \gamma, -\gamma) \\ & \times [f_1 b_-(\lambda + \lambda_0, \gamma, -\gamma) + 2 f_2 b_-(\lambda - \lambda_0, \gamma, \gamma)]. \end{aligned} \tag{12}$$

The next step is to seek for a suitable representation of the monodromy matrix. Previous experience in dealing with the quantum inverse-scattering method for genuine four-dimensional representation [10] suggests we take the following form

$$\mathcal{T}(\lambda) = \begin{pmatrix} B(\lambda) & \vec{B}(\lambda) & F(\lambda) \\ \vec{C}(\lambda) & \hat{A}(\lambda) & \vec{B}^*(\lambda) \\ C(\lambda) & \vec{C}^*(\lambda) & D(\lambda) \end{pmatrix}_{4\times 4} \tag{13}$$

where as in [10] $\vec{B}(\lambda)$, $\vec{B}^*(\lambda)$, $\vec{C}(\lambda)$ and $\vec{C}^*(\lambda)$ are two-dimensional vectors, the field $\hat{A}(\lambda)$ is a 2×2 matrix and the remaining operators are scalars.

Comparison between the above representation and the triangular property (9) reveals that $\vec{B}(\lambda)$, $\vec{B}^*(\lambda)$ and $F(\lambda)$ play the role of creation operators with respect to our choice of reference state. This is by no means an indication that all of them are needed to generate the eigenstates of the transfer matrix. To make further progress we have to investigate the commutation relations between the monodromy operators from the Yang–Baxter algebra (7). In order to present the results as generally as possible we find it convenient to represent the R-matrix by $R(\lambda, \mu) = \sum_{a,b,c,d}^{4} R_{ab}^{cd}(\lambda, \mu) e_{ac} \otimes e_{bd}$ where e_{ab} are the Weyl matrices. We first look at the commutation rules between the creation operator $\vec{B}(\lambda)$ with itself and with the diagonal fields and the results are

$$\vec{B}(\lambda) \otimes \vec{B}(\mu) = \frac{R_{22}^{22}(\lambda, \mu)}{R_{11}^{11}(\lambda, \mu)} [\vec{B}(\mu) \otimes \vec{B}(\lambda)] \cdot \hat{r}(\lambda, \mu) \tag{14}$$

$$B(\lambda)\vec{B}(\mu) = \frac{R_{11}^{11}(\mu, \lambda)}{R_{12}^{21}(\lambda, \mu)} \vec{B}(\mu) B(\lambda) - \frac{R_{21}^{21}(\mu, \lambda)}{R_{12}^{21}(\lambda, \mu)} \vec{B}(\lambda) B(\mu) \tag{15}$$

$$\hat{A}(\lambda) \otimes \vec{B}(\mu) = \frac{R_{22}^{22}(\lambda, \mu)}{R_{12}^{21}(\lambda, \mu)} [\vec{B}(\mu) \otimes \hat{A}(\lambda)] \cdot \hat{r}(\lambda, \mu) - \frac{R_{12}^{12}(\lambda, \mu)}{R_{12}^{21}(\lambda, \mu)} \vec{B}(\lambda) \otimes \hat{A}(\mu) \tag{16}$$

$$D(\lambda)\vec{B}(\mu) = \frac{R_{24}^{42}(\lambda, \mu)}{R_{13}^{31}(\lambda, \mu)} \vec{B}(\mu) D(\lambda) + \frac{R_{42}^{42}(\lambda, \mu)}{R_{13}^{31}(\lambda, \mu)} F(u) \vec{C}^{*}(\lambda) - \frac{R_{14}^{14}(\lambda, \mu)}{R_{13}^{31}(\lambda, \mu)} F(\lambda) \vec{C}^{*}(\mu). \tag{17}$$

The auxiliary matrix $\hat{r}(\lambda, \mu)$ is given by

$$\hat{r}(\lambda, \mu) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & a(\lambda, \mu) & b(\lambda, \mu) & 0 \\ 0 & b(\lambda, \mu) & a(\lambda, \mu) & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{18}$$

whose non-trivial matrix elements $a(\lambda, \mu)$ and $b(\lambda, \mu)$ are

$$a(\lambda, \mu) = \frac{\frac{2}{\cosh(2\gamma)}}{\frac{2}{\cosh(2\gamma)} + \cos(2\lambda) - \cos(2\mu)} \qquad b(\lambda, \mu) = -\frac{\cos(2\lambda) - \cos(2\mu)}{\frac{2}{\cosh(2\gamma)} + \cos(2\lambda) - \cos(2\mu)}. \tag{19}$$

From these results we first recognize that the matrix $\hat{r}(\lambda, \mu)$ is related to the one of the isotropic six-vertex model via reparametrization of the weights in terms of a new variable $\tilde{\lambda} = \cos(2\lambda)$. Next we observe from the commutation rules (14)–(16) that two creation fields $\vec{B}(\lambda)$ cannot generate other creation fields and also that all the unwanted terms coming from the diagonal fields $B(\lambda)$ and $\hat{A}(\lambda)$ remain in the space of the states of the operator $\vec{B}(\lambda)$. It is notable that such commutation relations are in the same form as that appearing in the nested Bethe ansatz solutions of multistate generalizations of the six-vertex model [11, 12]. A relevant difference, however, is the presence of the diagonal field $D(\lambda)$ which requires further elaboration. As far as the one-particle state is concerned, the commutation relation (16) shows that this operator generates unwanted terms that are trivially cancelled out due to the annihilation property $\vec{C}^{*}(\lambda)|0\rangle = 0$. To show that this is indeed a general situation we need the help of an extra commutation rule, namely

$$\vec{C}^{*}(\lambda) \otimes \vec{B}(\mu) = \frac{R_{22}^{22}(\lambda, \mu)}{R_{13}^{31}(\lambda, \mu)} \hat{r}(\lambda, \mu) \cdot \vec{B}(\mu) \otimes \vec{C}^{*}(\lambda) - \frac{R_{14}^{14}(\lambda, \mu)}{R_{13}^{31}(\lambda, \mu)} \vec{B}(\lambda) \otimes \vec{C}^{*}(\mu). \tag{20}$$

Now by iterating this relation on the tensor product of several $\vec{B}(\lambda_i)$ we are able to derive the following remarkable annihilation property

$$\vec{C}^{*}(\lambda) \otimes \vec{B}(\lambda_1) \otimes \cdots \otimes \vec{B}(\lambda_n)|0\rangle = 0 \tag{21}$$

which together with the commutation rule (17) implies that the field $D(\lambda)$ does not produce unwanted terms in the space of states generated by the creation fields $\vec{B}(\lambda)$.

These observations strongly suggest we take as the eigenvectors of the transfer matrix $T(\lambda)$ the following linear combination

$$|\Phi(\lambda_1, \ldots, \lambda_n)\rangle = \vec{B}(\lambda_1) \otimes \vec{B}(\lambda_2) \otimes \cdots \otimes \vec{B}(\lambda_n) \cdot \vec{\mathcal{F}}|0\rangle \tag{22}$$

where the vector $\vec{\mathcal{F}}$ depends on the parameter λ_i and its components will be denoted by $\mathcal{F}^{a_1\cdots a_n}$ where each $a_i = 1, 2$.

At this point we have gathered the basic tools to determine the eigenvalues $\Lambda(\lambda, \{\lambda_i\})$ of the transfer matrix defined by

$$\left[B(\lambda) + \sum_{a=1}^{2} A_{aa}(\lambda) + D(\lambda)\right]|\Phi(\lambda_1, \ldots, \lambda_n)\rangle = \Lambda(\lambda, \{\lambda_i\})|\Phi(\lambda_1, \ldots, \lambda_n)\rangle. \tag{23}$$

In order to do that we use the commutation rules (15)–(17) and the property (21) to commute the diagonal operator through all the $\vec{B}(\lambda_i)$ fields until they hit the reference state. Their values acting on the reference state can be read out from equation (9). Omitting details of the calculations our final result for the eigenvalues of the transfer matrix is

$$\begin{aligned}\Lambda(\lambda, \{\lambda_j\}) = [\omega_1(\lambda)]^N \prod_{j=1}^{n} \frac{R_{11}^{11}(\lambda_j, \lambda)}{R_{12}^{21}(\lambda_j, \lambda)} + [\omega_3(\lambda)]^N \prod_{j=1}^{n} \frac{R_{24}^{42}(\lambda, \lambda_j)}{R_{13}^{31}(\lambda, \lambda_j)} \\ + [\omega_2(\lambda)]^N \prod_{j=1}^{n} \frac{R_{22}^{22}(\lambda, \lambda_j)}{R_{12}^{21}(\lambda, \lambda_j)} \Lambda^{(1)}(\lambda, \{\lambda_l\})\end{aligned} \tag{24}$$

provided that the rapidities satisfy the Bethe equations

$$\left[\frac{\omega_1(\lambda_i)}{\omega_2(\lambda_i)}\right]^L = (-1)^{n-1}\Lambda^{(1)}(\lambda = \lambda_i, \{\lambda_j\}) \qquad i = 1, \ldots, n \tag{25}$$

The function $\Lambda^{(1)}(\lambda, \{\lambda_l\})$ refers to the eigenvalue of an auxiliary problem related to the condition that the vector $\vec{\mathcal{F}}$ is an eigenvector of the transfer matrix whose Boltzmann weights are the R-matrices (18), namely

$$\hat{r}_{b_1 d_1}^{c_1 a_1}(\lambda, \lambda_1)\hat{r}_{b_2 c_2}^{d_1 a_2}(\lambda, \lambda_2) \cdots \hat{r}_{b_n c_1}^{d_{n-1} a_n}(\lambda, \lambda_n)\mathcal{F}^{a_n \cdots a_1} = \Lambda^{(1)}(\lambda, \{\lambda_j\})\mathcal{F}^{b_n \cdots b_1}. \tag{26}$$

This problem is solved by another Bethe ansatz which gives the well-known results [8, 9] for the inhomogeneous rational six-vertex model

$$\Lambda^{(1)}(\lambda, \{\lambda_i\}, \{\mu_j\}) = \prod_{j=1}^{m} \frac{1}{b(\mu_j, \lambda)} + \prod_{i=1}^{n} b(\lambda, \lambda_i) \prod_{j=1}^{m} \frac{1}{b(\lambda, \mu_j)} \tag{27}$$

where the auxiliary variables $\{\mu_j\}$ are subject to the following set of constraints

$$\prod_{i=1}^{n} b(\mu_j, \lambda_i) = -\prod_{k=1}^{m} \frac{b(\mu_j, \mu_k)}{b(\mu_k, \mu_j)} \qquad j = 1, \ldots, m. \tag{28}$$

Now we first insert the nested eigenvalues (27) into equation (24). By taking into account the explicit expression of the R-matrix elements after carrying out cumbersome simplifications we find that

$$\begin{aligned}\Lambda(\lambda, \{\lambda_j\}, \{\mu_l\}) = [\omega_1(\lambda)]^L \prod_{j=1}^{n} z(\lambda, \gamma) \left[\frac{U - \cos(2\lambda_i) + \cos(2\lambda)}{\cos(2\lambda_i) - \cos(2\lambda)}\right] + [\omega_3(\lambda)]^L \prod_{j=1}^{n} z(\lambda, \gamma) \\ + [\omega_2(\lambda)]^L \left\{ \prod_{j=1}^{n} z(\lambda, \gamma) \left[\frac{U + \cos(2\lambda) - \cos(2\lambda_i)}{\cos(2\lambda) - \cos(2\lambda_i)}\right]\right.\end{aligned}$$

$$\times \prod_{l=1}^{m} \frac{U + \cos(2\mu_l) - \cos(2\lambda)}{\cos(2\lambda) - \cos(2\mu_l)}$$

$$+ \prod_{j=1}^{n} -z(\lambda, \gamma) \prod_{l=1}^{m} \left[\frac{U + \cos(2\lambda) - \cos(2\mu_l)}{\cos(2\mu_l) - \cos(2\lambda)} \right] \Bigg\} \tag{29}$$

where the function $z(\lambda, \gamma)$ and the parameter U are given by

$$z(\lambda, \gamma) = \begin{cases} \frac{b_+(\lambda,\gamma,0)}{a_-(\lambda,\gamma,0)} & \text{for} \quad B = 0 \\ -\frac{a_+(\lambda,\gamma,0)}{b_-(\lambda,\gamma,0)} & \text{for} \quad A = 0 \end{cases} \qquad \text{and} \qquad U = \begin{cases} \frac{2}{\cosh(2\gamma)} & \text{for} \quad B = 0 \\ -\frac{2}{\cosh(2\gamma)} & \text{for} \quad A = 0 \end{cases}. \tag{30}$$

Proceeding similarly for equations (25) and (28) we obtain the corresponding nested Bethe ansatz equations

$$\left[\frac{U - \cos(2\lambda_i) + \cos(2\lambda_0)}{-\cos(2\lambda_i) + \cos(2\lambda_0)} \right]^N [-z(\lambda_0, \gamma_0)]^N$$

$$= (-1)^{n-1} \prod_{l=1}^{m} \frac{U + \cos(2\mu_l) - \cos(2\lambda_i)}{\cos(2\lambda_i) - \cos(2\mu_l)} \qquad j = 1, \ldots, n \tag{31}$$

$$\prod_{j=1}^{n} \frac{\cos(2\lambda_j) - \cos(2\mu_l)}{U + \cos(2\mu_l) - \cos(2\lambda_j)} = -\prod_{k=1}^{m} \frac{\cos(2\mu_l) - \cos(2\mu_k) - U}{\cos(2\mu_l) - \cos(2\mu_k) + U} \qquad l = 1, \ldots, m.$$

From the result (31) we observe that it is possible to cast the Bethe equations in a simpler form if we define the following new rapidities

$$\frac{\cos(2\lambda_i)}{U} = \tilde{\lambda}_i + \frac{1}{2} + \frac{\cos(2\lambda_0)}{U} \qquad \frac{\cos(2\mu_i)}{U} = \tilde{\mu}_i + \frac{\cos(2\lambda_0)}{U} \tag{32}$$

leading us to the expressions

$$\begin{aligned} &\left[\frac{\tilde{\lambda}_i - \frac{1}{2}}{\tilde{\lambda}_i + \frac{1}{2}} \right]^N [-z(\lambda_0, \gamma_0)]^N = (-1)^{n+m-1} \prod_{l=1}^{m} \frac{\tilde{\lambda}_i - \tilde{\mu}_l - \frac{1}{2}}{\tilde{\lambda}_i - \tilde{\mu}_l + \frac{1}{2}} \qquad j = 1, \ldots, n \\ &(-1)^n \prod_{j=1}^{n} \frac{\tilde{\mu}_l - \tilde{\lambda}_j - \frac{1}{2}}{\tilde{\mu}_l - \tilde{\lambda}_j + \frac{1}{2}} = -\prod_{k=1}^{m} \frac{\tilde{\mu}_l - \tilde{\mu}_k - 1}{\tilde{\mu}_l - \tilde{\mu}_k + 1} \qquad l = 1, \ldots, m. \end{aligned} \tag{33}$$

In terms of these variables the dependence of the Bethe equations on the free-parameters becomes restricted as a fixed phase factor. Next we observe that their rational form resembles that of the supersymmetric *t–J* model [13, 14] with the important difference that in our case we have a bigger space of states such that $m \leqslant N$ rather than $m \leqslant N/2$. As far as the Hilbert space is concerned one would be tempted to compare these equations with that of the supersymmetric U model [15]. However, this demands that the continuous parameter governing the four-dimensional representation of the superalgebra $spl(2|1)$ reaches the value related to atypical representations which is not permitted in this model [16, 17]. These remarks are strong evidence that the class of models solved here lies in between these two supersymmetric systems motivating subsequent investigations. One of them consists in the computation of physical properties of the bilayer vertex model such as the ground state behaviour dependence on the free-parameters. Another interesting point raised here is the possible connection between this system and four-dimensional atypical $spl(2|1)$ representations. It also seems worthwhile to generalize our results to include the manifold discovered by Shiroishi and Wadati for arbitrary values of the constants A and B. It is plausible to believe that this will lead us to Bethe equations interpolating between those of the t–J and Hubbard-like models. In this situation a crucial point is to unveil the corresponding six-vertex hidden symmetry which has eluded us so far.

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